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Configuration Ray Representations in Non-Abelian Quantum Kinematics and Dynamics

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Ray extensions of the regular representation of noncompact non-Abelian Lie groups are examined as generalizations of the Cartesian coordinate representation of ordinary quantum mechanics to the case of generalized non-Cartesian coordinates and generalized noncommuting momenta. (The momenta are in fact the generators of the representation, and so they satisfy the Lie algebra of the group.) The concept of configuration ray representation is introduced within this new kinematic formalism as subrepresentations of the regular representation which are embossed with the "relativity theory" of a given system. The main features of the mathematical formalism leading to these representations in configuration spacetime are discussed, and their importance for non-Abelian quantum kinematics and dynamics is emphasized. Two miscellaneous examples on the calculus of phase functions for configuration ray representations are given.

1. INTRODUCTION

In a previous paper (Krause, 1987; hereafter referred as paper I) we have presented a rather simple formalism of 2-cocycle calculus for unitary ray representations of Lie groups. This formalism may be especially attractive to physicists, in general, since it can be used in quantum theories without requiring a working knowledge of cohomology (Michel, 1964). In this paper we wish to examine this matter further, from the special standpoint of its applications in non-Abelian quantum kinematics and dynamics.

Although the method used in paper I is not specific to a particular unitary representation, the analysis and calculations made in that paper were anchored on the group manifold. The group manifold is usually not the carrier space of the relevant realizations of Lie groups in physics.

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Nevertheless, as a matter of fact, our approach to 2-cocycles is general enough and means no restriction on the *method* of exponent factor calculus (see also Houard, 1977).

The same approach has been used recently for studying a possible connection between non-Abelian quantum kinematics and dynamics for noncompact Lie groups (Krause, 1993*c*). However, a systematic, general, and detailed discussion of the *mathematical tools* one need to use in order to apply the ideas of quantum kinematics to quantum dynamics is still missing in the literature. We devote this paper to filling this gap.

So, the present work has a purely instrumental character. It bears some interest for physics, however, for it presents in a simple fashion new powerful group-theoretic techniques which may have many useful applications in quantum theories (see, for instance, Krause, 1986, 1988). For the physical motivation of non-Abelian quantum kinematics, in general, we refer the reader to our previous work (Krause, 1993*b*, and references quoted therein).

There are two well-known facts we would like to recall concerning this issue. First, the functions $g(q'; q)$ (which combine the parameters q' and q according to the composition law of the group) may themselves be considered as defining two separate point symmetry groups (the *first-parameter group* G_1 and the *second-parameter group* G_2), which are in fact isomorphic with the original group G because they have the same group manifold $M(G) = \{q\}$ and because the laws of composition are essentially the same for G_1, G_2, and G. Strictly speaking, G_2 is antiisomorphic with G, since it is isomorphic when the group elements are taken in the reverse order. [For these and other details see Racah (1965).] Moreover, the group manifold itself is the homogeneous carrier space of these self-realizations of G; so one does not need to consider an extraneous space for visualizing the action of the group.

The other fact concerns the regular representation of Lie groups, which is a subject that has been amply studied by mathematicians (Naimark and Stern, 1982). Indeed, the regular representation has the peculiarity of being another self-contained faithful construct whose building blocks belong exclusively to the group structure itself, since the carrier Hilbert space $\mathscr{H}(G)$ is defined on the group manifold $M(G)$ by means of the invariant (right and left) measures obtained from the composition law $g(q'; q)$. This feature makes the regular representation an outstanding group-theoretic notion. Because of this same feature, however, the regular representation has not yet played an explicit role in quantum theory, since it appears as a formalism that lies far away from the concrete physical pictures on which Lie groups operate.

Notwithstanding this fact, there is one good reason to choose the regular representation as the cornerstone of quantum kinematics if one wants to generalize the canonical quantization rule (Komar, 1971) beyond the Heisenberg commutation relations that have been used hitherto in ordinary quantum mechanics (Krause, 1985, 1993*c*). Obviously, Heisenberg's quantum kinematics (Weyl, 1931) is not valid in the case of generalized momenta satisfying a non-Abelian Lie algebra (Yamada, 1982). [Angular momenta are not an exception to this contingency, because in quantum mechanics one does not "quantize" the canonical conjugate variables of non-Abelian momenta (e.g., Euler angles).] Now, the point is that the standard Heisenberg commutation relations are intimately related with the *regular representation* of the Abelian group of rigid translations in a Cartesian affine manifold; in this sense they appear as a *necessary and sufficient* construct indeed (see, e.g., Messiah, 1961). Hence, the well-known dictum: "one must quantize only in Cartesian coordinates" follows. Certainly, this is not a satisfactory state of affairs (Komar, 1971). As a matter of fact, we see that the regular representation plays an outstanding role in ordinary quantum mechanics, at least in that unique case corresponding to Heisenberg's Abelian quantum kinematics of rigid Cartesian translations, which does not correspond to a universal symmetry, however (Krause, 1993*c*).

It is clear that the (left and right) regular representations of Lie groups are intimately related with the (left and right) self-realizations G_1 and G_2 mentioned above. These two self-contained structures settle the theoretical framework for having non-Abelian 'wave mechanics' over the group manifold arena (Krause, 1985). To this end, one *quantizes the group* associating its parameters q^a with a set of commuting Hermitian operators Q^a, which act within the rigged Hilbert space $\tilde{\mathscr{H}}(G)$ that carries the regular representation. The Q's are generalized 'position' operators of the group manifold; they yield the 'Q-representation' of G in $\mathscr{H}(G)$, even if the group is non-Abelian. On the other hand, the generators of the representation are Hermitian operators on their own right. Thus one obtains *generalized Heisenberg commutation relations* as well as a *generalized Heisenberg–Weyl quantum-kinematic (closed) algebra*. Furthermore, it can be shown in this manner that *every* r-dimensional Lie group has a set of r *quantum-kinematic invariant operators*, which substantially differ from the traditional invariant operators of Lie algebras and their enveloping algebras (Krause, 1991). Also, *boson ladder operators* have been found, quite generally, within the quantum-kinematic formalism of noncompact Lie groups (Krause, 1993*a*), and their associated coherent states have been discussed recently (Krause, 1993*b*). So one finds enough structure in order to build a rich non-Abelian quantum kinematics which generalizes the standard Heisenberg kinematics in a rather natural way.

To be sure, in this fashion one expects to obtain meaningful dynamical quantum models if G is a physically relevant Lie group (Krause, 1993*c*).

However, one has to recognize that in the applications of unitary ray representations to quantum kinematics, 2-cocycles anchored on $M(G) \times M(G)$ are contrived to produce quantum models which look very different from ordinary wave mechanical (e.g., Schrödinger) models; therefore their physical interpretation becomes blurred. Certainly, quantum kinematics needs a configuration description of the model, which should be concordant with the 'special relativity' theory of the system one wants to quantize (Mariwalla, 1975). Lie groups often appear in physics as point symmetry transformations of some preassigned space which has a clear physical significance. One refers to such realizations of Lie groups as *geometric realizations* (Olver, 1986). Accordingly, if non-Abelian quantum kinematics is able to afford a new foundation of quantum mechanics, we should have a general prescription for settling a ray representation which is directly embossed on any given geometric realization of a Lie group acting in the configuration space of a system. Thus we could attain a quantum formalism which would be intrinsically 'relativistic' in a very ample sense (see also Krause, 1993*c*).

Specifically, this paper is devoted to the task of developing such a formalism of *configuration ray representations* of noncompact Lie groups under the assumption that these are subrepresentations of the regular ray representation. To this end we here follow the same method already employed in paper I, but now G acts on a given homogeneous space X, corresponding to the configuration space of a G-invariant system.

The contents of this paper are as follows. In Section 2 we begin our study of configuration ray representations of a given Lie group. This section is devoted mainly to presenting a very simple technique for calculating the required exponents (i.e., phase functions) without using cohomological notions, which are reserved to specialists and remain unknown to most physicists. Section 3 is a short study of the gauge freedom of configuration ray representations. In Section 4, the generating wave function of a configuration ray representation is considered. (From the standpoint of quantum kinematics, this is the heart of the matter.) Section 5 contains two miscellaneous examples of non-Abelian phase calculus. Finally, in Section 6 we add some concluding remarks concerning quantum kinematics and dynamics.

In this paper we shall not repeat the general formalism of non-Abelian quantum kinematics, which has already been developed in our previously published work on this subject. Rather, we assume that the reader has some familiarity with that work [in particular, Krause (1993*b*, Section 2) is enough to this end]. Our main reference, however, is paper I, of which the present paper is a necessary complement.

2. CONFIGURATION RAY REPRESENTATIONS

Let us consider the central extensions by $U(1)$ of quantum kinematics for noncompact non-Abelian Lie groups. It is a well-known fact that quantum mechanics does not fix the phase of the vectors describing pure states, and one associates such states to rays rather than to vectors. Therefore, unitary ray representations should be used in quantum kinematics in general. The extension of the unitary formalism of non-Abelian quantum kinematics from "true" (vector) to "projective" (ray) representations faces no difficulties as long as one is able to calculate the admissible two-cocycles of the corresponding group.

The notation used in this paper is the same as used in our previous work. Henceforth, G denotes a noncompact, connected and simply connected, r-dimensional non-Abelian Lie group, with the following global property: there exists a coordinate system $q = (q^1, \ldots, q^r)$ that covers the whole group manifold $M(G)$ and maintains everywhere a one-to-one correspondence with the elements of G; namely, the coordinates q^a, $a = 1, \ldots, r$, are real and provide a set of r essential parameters of G. There are many physically interesting noncompact Lie groups for which this simplifying assumption holds. (Quantum kinematics, however, is a formalism that is independent of this special assumption.)

In quantum kinematics, G establishes an isomorphism between rays that preserves all transition probabilities. Therefore, it is useful to define unitary (or antiunitary) operator rays, in analogy to the notion of vector rays. In this fashion, according to Wigner's theorem, the operators of the isomorphism are representatives selected from the corresponding operator rays (Wigner, 1959). Hence, one infers (by well-known arguments) the ray representation property:

$$U_L^{(k)}(q)U_L^{(k)}(q') = e^{i\phi_k(q;q')}U_L^{(k)}[g(q;q')] \tag{2.1}$$

for the central extensions [by $U(1)$] of the *left regular representation* of G. The two-cocycle $\phi_k(q;q')$ is a real-valued function defined globally on $M(G) \times M(G)$, since G is connected and simply connected (i.e., Bargmann's theorem); it is also differentiable everywhere, having continuous partial derivatives of all orders on $M(G) \times M(G)$ (e.g., Iwasawa's theorem). Furthermore, since G is connected, it can be shown that the operators $U_L^{(k)}(q)$ are necessarily unitary. The same features hold for the right regular ray representation. [For these and other details, see Bargmann (1954).] Right now, k is a label for denoting the selection of the representative unitary operators. As we have already said, in equation (2.1) $g(q;q')$ stands for the group-multiplication functions in the adopted parametrization of G.

Now, the associative property of the representation (2.1) yields the following well-known functional relation for the exponent function:

$$\phi_k(q; q') + \phi_k[g(q; q'); q''] = \phi_k(q'; q'') + \phi_k[q; g(q'; q'')] \qquad (2.2)$$

This three-point functional relation represents the backbone of two-cocycle analysis (cf. paper I). In the applications, it is useful to consider two-cocycles $\phi_k(q; q')$ which satisfy the special gauge condition

$$\mu_k(q) \equiv \phi_k(q; \bar{q}) = \phi_k(\bar{q}; q) = 0 \qquad (2.3)$$

for all $q \in M(G)$, for then it follows that

$$U_L^{(k)\dagger}(q) = U_L^{(k)}(\bar{q}) \qquad (2.4)$$

In fact, the μ-gauge (2.3) brings many nice simplifications into the formalism. We shall assume this gauge in what follows.

Now, let $X = \{x\}$ be the configuration spacetime of a system (Trümper, 1983), and let G be a Lie group of kinematical automorphisms acting transitively on X. (For the sequel, it is sufficient to think of X as a homogeneous space of G.) The action of G on the events $x = (x^0, x^1, \ldots, x^n)$ is realized by a set of point transformations, say

$$x'^\mu = f^\mu(x; q) \qquad (2.5)$$

$\mu = 0, 1, \ldots, n$. Note that the x^μs are *not* Cartesian coordinates in general. One writes the group law as usual:

$$f^\mu[f(x; q); q')] = f^\mu[x; g(q'; q)] \qquad (2.6)$$

$$f^\mu[f(x; q); g(q'; q'')] = f^\mu\{f[x; g(q''; q)]; q'\} \qquad (2.7)$$

In this manner, the action of G on the points $x \in X$ becomes realized by a set of point transformations $f(G)\colon X \to X$ given by equation (2.5), with $q \in M(G)$, and such that $q \neq e \Rightarrow x' \neq x$. The functions f^μ are analytic real functions of the q's which are endowed with the group property of G. Thus $x^\mu = f^\mu(x; e)$ holds for all $x \in X$. The "intrinsic associativity" (2.7) of the f's is a consequence of the group law (2.6).

In non-Abelian quantum kinematics we give up the canonical quantization procedure, following a different path of approach to quantum dynamics. (Our approach is in fact more akin with quantum field theory.) In order to obtain a *configuration quantum model* of a system that manifests the action of the symmetry group G on X, we do *not* quantize the preferred coordinates (i.e., we give up the usual association $x^\mu \to X^\mu$ to begin with). Instead, we pose the problem of finding kets $|x\rangle = |x^0, \ldots, x^n\rangle \in \mathscr{H}(G)$ (rigged) which are in one-to-one correspondence with $x = (x^0, \ldots, x^n) \in X$ and which under the action of the *left* unitary ray operators of G transform in a covariant manner with respect to equation (2.5); namely, we require

$$U_L^{(k)}(q)|x\rangle = e^{i\varphi_k(x;q)}|f(x;q)\rangle \tag{2.8}$$

where the exponent $\varphi_k(x;q)$ is a real-valued phase function. We say that such kets carry a *configuration ray representation* of G on X. (Henceforth we adopt the *left* regular representation as our working frame.) In order to tackle the problem set by the existence of such configuration kets in $\tilde{\mathscr{H}}(G)$, one needs to search for *necessary and sufficient* conditions for the property announced in equation (2.8) to hold.

Let us first examine the phase function. Besides the obvious "initial condition"

$$\varphi_k(x;e) = 0 \tag{2.9}$$

for all $x \in X$, it is easy to see that it has to satisfy the functional relation

$$\varphi_k(x;q) + \varphi_k[f(x;q);q'] - \varphi_k[x;g(q';q)] = \phi_k(q;q') \tag{2.10}$$

One can check the consistency of equation (2.10) with equation (2.2). It will be shown below that, given the function $f^\mu(x;q)$ and $\phi_k(q';q)$, one has enough information for getting a consistent method in order to calculate admissible phase functions.

Two important consequences are immediate. We note that the regular ray representation of G requires a configuration *ray* representation; we also note that for a nontrivial ray representation of G the phase $\varphi_k(x;q)$ *must* be a function of x (otherwise, ϕ_k would be a coboundary). By the same token, one has

$$\varphi_k(x;q) + \varphi_k[f(x;q);\bar{q}] = \mu_k(q) = 0 \tag{2.11}$$

since we here assume the μ-gauge.

We next obtain differential equations for the phase functions $\varphi_k(x;q)$, using the same method as introduced in paper I. Thus, we first define a set of *phase generators*; namely

$$\sigma_a^{(k)}(x) = \lim_{q \to e} \partial_a \varphi_k(x;q) \tag{2.12}$$

for $a = 1, \ldots, r$. So performing the limits $\lim_{q' \to e} \partial'_a$ and $\lim_{q \to a} \partial_a$ on equation (2.10), we obtain the following differential equations for the phase function $\varphi_k(x;q)$:

$$X_a(q)\varphi_k(x;q) = \sigma_a^{(k)}[f(x;q)] - r_a^{(k)}(q) \tag{2.13}$$

$$[Y_a(q) - Z_a(x)]\varphi_k(x;q) = \sigma_a^{(k)}(x) - l_a^{(k)}(q) \tag{2.14}$$

where $Z_a(x)$ are Lie differential operators on X; e.g., $Z_a(x) = u_a^\mu(x)\partial_u$, with $u_a^\mu(x) = f^\mu_{,a}(x; q)|_{q=e}$. Here we have also used the definitions of the *right* and *left exponent generators* $r_a^{(k)}(q)$ and $l_a^{(k)}(q)$ given by $r_a^{(k)}(q) = \partial'_a\phi_k(q'; q)|_{q'=e}$ and $l_a^{(k)}(q) = \partial'_a\phi_k(q; q')|_{q'=e}$, respectively. In the same way, one defines Lie (right and left) vector fields as $X_a(q) \equiv R_a^b(q)\partial_b$ and $Y_a(q) \equiv L_a^b(q)\partial_b$, where R_a^b and L_a^b are the (right and left) *transport matrices* for contravariant vectors in $M(G)$, which are obtained from $g^a(q; q)$ in the usual fashion; i.e., $R_a^b(q) = \partial'_a g^b(q'; q)|_{q'=e}$ and $L_a^b(q) = \partial'_a g^b(q; q')|_{q'=e}$. Finally, from equations (2.13), one obtains the following *inhomogeneous non-Abelian curl equations* for the phase generators:

$$Z_a(x)\sigma_b^{(k)}(x) - Z_b(x)\sigma_a^{(k)}(x) - f^c_{ab}\sigma_c^{(k)}(x) = -k_{ab} \tag{2.15}$$

where k_{ab} corresponds to the *ray constants* of the representation (Bargmann, 1954).

Let us then consider the *converse* problem in order to find a synthetic (i.e., constructive) standpoint for phase calculus on $X \times M(G)$. We first observe that equation (2.15) entails a system of linear differential equations which must be satisfied by the phase generators $\sigma_a^{(k)}(x)$ in order to have a consistent solution at all. In fact, it can be proved that if one solves equation (2.15) and then uses $\sigma_a^{(k)}(x)$ as sources to solve equations (2.13) and (2.14) with the initial condition (2.9), one finds a phase function $\varphi_k(x; q)$ which automatically satisfies the functional relation (2.10). So we have the following result.

Theorem. Given a 2-cocycle $\phi_k(q'; q)$, a necessary and sufficient condition for the 2-point functional relation (2.10) to hold is that the phase function $\varphi_k(x; q)$ satisfies equations (2.13) and (2.14) with the homogeneous initial condition (2.9).

We observe that no initial condition at some specified point $x_0 \in X$ is required to solve these equations. The reason for this is simple: as a consequence of the definition (2.12) and of the initial condition (2.9), together with the property (cf. paper I):

$$r_a^{(k)}(e) = l_a^{(k)}(e) = 0 \tag{2.16}$$

the differential equations (2.13) and (2.14) are in fact *equivalent*, and therefore the initial condition (2.9) at $q = e$ is enough for solving these equations.

Hence, given a regular ray representation of G, the *method* for having an allowable phase function of a configuration ray representation is clear. Namely (1) using the given set of ray constants, one solves equation (2.15) for the phase generators; (2) next, using the generators $\sigma_a^{(k)}(x)$ and $r_a^{(k)}(q)$ [or $l_a^{(k)}(q)$] as sources, one solves equation (2.13) [or, for that matter, equation (2.14)] with the initial condition (2.9); and (3) one thus obtains a phase function φ_k which satisfies the 2-point functional relation (2.10). (However, the solution is not unique, as we shall discuss presently.)

Finally, let us also briefly consider the possibility of having a configuration *ray* extension associated with the regular *vector* representation of G. In this case, instead of equation (2.15), one has to solve the *homogeneous* non-Abelian curl equations:

$$Z_a(x)\sigma_b(x) - Z_b(x)\sigma_a(x) - f^c_{ab}\sigma_c(x) = 0 \tag{2.17}$$

It can be shown that the linearly independent solutions to these equations are necessarily of the trivial form

$$\sigma_a(x) = Z_a(x)\lambda(x) \tag{2.18}$$

where λ is a function of x that remains undetermined. If one substitutes this solution into equation (2.13) [with $r_a(q) \equiv 0$] one gets

$$X_a(q)\varphi(x; q) = Z_a[f(x; q)]\lambda[f(x; q)] = X_a(q)\lambda[f(x; q)] \tag{2.19}$$

from which the phase function

$$\varphi(x; q) = \lambda[f(x; q)] - \lambda(x) \tag{2.20}$$

follows. When this expression for $\varphi(x; q)$ is substituted into equation (2.14) [with $l_a(q) \equiv 0$] it yields an identity indeed. One easily checks the functional relation (2.10) when $\varphi(x; q)$ is of the form (2.20) and $\phi = 0$.

The phase function $\varphi(x; q)$ that figures equation (2.20) corresponds to a *coboundary* of G in $X \times M(G)$. The interest of having configuration ray extensions attached to the "true" (i.e., vector) regular representation is not purely academic. In fact, it is well known that many physically relevant Lie groups do *not* admit genuine central ray extensions by $U(1)$, because the associated Lie algebras are such that some ray constants are necessarily zero ($k_{ab} = 0$), while the others are all trivial (i.e., they are coboundaries of the Lie algebra of the form $k_{ab} = f^c_{ab}k_c$) (cf. paper I). All compact Lie groups [like all $SU(n)$, for instance] share this property; all its 2-cocycles $\phi(q; q')$ are just coboundaries in $M(G) \times M(G)$, so they are gauge artifacts and, as such, they can be eliminated (cf. next section). The most conspicusous example of a noncompact Lie group having this property is the Poincaré group $\mathscr{P}^\uparrow_+(1, 3)$ [though not so the Poincaré group $\mathscr{P}^\uparrow_+(1, 1)$ in 2-dimensional Minkowski space]. However, this is not to say that a coboundary $\varphi(x; q)$ in $X \times M(G)$ is useless in quantum kinematics. To the contrary, such phase functions stem from the direct product $U(1) \times G$, since they correspond to a *local* action of $U(1)$ on the configuration kets $|x\rangle$ associated with the action of G on X (cf. below). Hence, they can be as important in quantum kinematics as they are in gauge field theories (see, e.g., Chaichian and Nelipa, 1984), especially so in relation with the electromagnetic interaction, which is a well-known fact.

Clearly, in some special instances, for the case of the regular vector representation of G one can set $\varphi(x; q) = 0$ without loss of generality, and thus it is also possible to adjust a configuration *vector* representation within $\mathscr{H}(G)$.

3. GAUGE TRANSFORMATIONS OF CONFIGURATION RAY REPRESENTATIONS

The 2-cocycle used in a given representation corresponds to the analytical expression of the selection of representatives from the unitary ray operators. Hence, if one changes the selection, one gets a gauge transformation (of the second kind) of the general form

$$U_L^{(k')}(q) = \{\exp[i\lambda_{k'k}(q)]\} U_L^{(k)}(q) \tag{3.1}$$

[with $\lambda_{k'k}(e) = 0$]. Thus, one gets a new exponent function, given by

$$\phi_{k'}(q'; q) = \phi_k(q'; q) + \lambda_{k'k}(q') + \lambda_{k'k}(q) - \lambda_{k'k}[g(q'; q)] \tag{3.2}$$

which also satisfies equation (2.2). The generators of the new 2-cocycle are then given by

$$\begin{aligned} r_a^{(k')}(q) &= r_a^{(k)}(q) - X_a(q)\lambda_{k'k}(q) + k_a \\ l_a^{(k')}(q) &= l_a^{(k)}(q) - Y_a(q)\lambda_{k'k}(q) + k_a \end{aligned} \tag{3.3}$$

and thus a set of new ray constants follows:

$$k'_{ab} = k_{ab} + f^c_{ab} k_c \tag{3.4}$$

where we have written $k_a = \lambda_{k'k,a}(e)$. Two exponents related in this fashion are called *equivalent*. Hence, one has the familiar result: *every unitary ray representation of G defines in a unique way only a class of equivalent 2-cocycles*. The following feature is also interesting: a general gauge transformation of the ray constants as stated in equation (3.4), with k_a arbitrary, is a necessary and sufficient condition for obtaining a gauge transformation of the exponent function as given in equation (3.1). Let us briefly refer to these gauge transformations as *λ-transformations*. In equation (3.1) one assumes a transformation $k_{ab} \to k'_{ab}$ of the ray constants in general. However, in the special case $k_a = 0$ (i.e., $k'_{ab} = k_{ab}$), one has a *restricted* λ-transformation. (For details, see paper I.)

Let us also recall that *trivial* unitary ray representations are those whose exponent functions are *coboundaries* of the form

$$\phi(q'; q) = \lambda[g(q'; q)] - \lambda(q') - \lambda(q) \tag{3.5}$$

They are nothing but gauge artifacts. [In the jargon of cohomology theory, genuine extensions of G by $U(1)$ are characterized by the second cohomology group $H_0^2[G, U(1)]$ (i.e., cocycles/coboundaries)]. So, *trivial exponent generators* are given by

$$r_a(q) = X_a(q)\lambda(q) - k_a, \qquad l_a(q) = Y_a(q)\lambda(q) - k_a \tag{3.6}$$

[here we set $k_a = \lambda_{,a}(e)$], and therefore the corresponding *trivial ray constants* are given by $k_{ab} = f^c_{ab}k_c$, as we have already mentioned. It is clear that a necessary and sufficient condition for having a trivial two-cocycle is that *all* the ray constants are trivial. Furthermore, it is also rather clear that one cannot eliminate a genuine (i.e., nontrivial) ray representation of G by means of a λ-transformation. [As has been said, trivial unitary ray representations of G correspond to the local *direct product* of G by $U(1)$; i.e., $U_L^{(\lambda)}(q) = e^{-i\lambda(q)}U_L(q)$.]

Finally, let us add a comment on the special μ-gauge defined in equation (2.3). In order to transform a given unitary ray representation into this gauge one uses $\lambda_k(q) = -\frac{1}{2}\mu_k(q)$ as generator of the required λ-transformation. However, one still has some remaining gauge freedom for selecting the representative operators within the μ-gauge. Indeed, this freedom entails the property $\lambda_k(q) + \lambda_k(\bar{q}) = 0$, which must be satisfied everywhere by the gauge generating function. Note that within the μ-gauge one has $\phi_k(q'; q) + \phi_k(\bar{q}; \bar{q}') = 0$ and $l_a^{(k)}(q) = r_a^{(k)}(\bar{q})$ (cf. paper I for details).

Our task is now to discuss the consequences of equation (3.1) within the configuration ray representation formalism. It is immediate that a λ-transformation induces a gauge transformation on the phase function of the configuration ray representation; namely

$$\varphi_{k'}(x; q) = \varphi_k(x; q) + \lambda_{k'k}(q) \tag{3.7}$$

Moreover, because of the induced gauge transformation (3.2) on $\phi_k(q'; q)$, the 2-point functional relation (2.10) is gauge invariant under λ-transformations (3.1) and (3.7). Furthermore, equation (3.7) produces the following transformation on the phase generators of the configuration ray representation:

$$\sigma_a^{(k')}(x) = \sigma_a^{(k)}(x) + k_a \tag{3.8}$$

where one defines the constants $k = \lambda_a^{(k'k)} = \partial_a \lambda^{(k;k)}(q)\big|_{q=e}$ as in equation (3.4).

One can prove in fact that configuration ray representations are gauge covariant under λ-transformations. Indeed, equations (2.10) and (2.13)–(2.15) all become covariantly transformed upon (3.1). (In the applications, one is mainly interested in restricted λ-transformations.)

Within the configuration ray representation formalism, one also has some gauge freedom for fixing the phase of the kets $|x\rangle$ themselves locally in space X; i.e., one sets

$$|x\rangle \to |x\rangle_\lambda = e^{i\lambda(x)}|x\rangle \tag{3.9}$$

This change of gauge is completely independent of the gauge one adopts for the regular ray representation. Of course, λ is an arbitrary function of x. As a consequence of equation (3.9), one has to change the phase function accordingly; namely, one sets

$$\varphi'_k(x; q) = \varphi_k(x; q) + \lambda(x) - \lambda[f(x; q)] \tag{10}$$

where, clearly, $\varphi'_k(e; x) = \varphi_k(e; x) = 0$. The functional relation (2.10), as well as equation (2.11), remain invariant under these transformations. From equation (3.10) one obtains

$$\sigma'^{(k)}_a(x) = \sigma^{(k)}_a(x) - Z_a(x)\lambda(x) \tag{3.11}$$

which states the corresponding transformation law for the phase generators [e.g., equation (2.18)].

In this fashion, it can be shown that the whole formalism of phase calculus for configuration ray representations remains invariant under transformations (3.9). Thus the solutions to the differential equations (2.13) and (2.14) are defined only up to a local $U(1)$ transformation as defined in equation (3.9). Again, even in the case when G admits genuine ray representations, these arbitrary local $U(1)$ transformations of the associated configuration *ray* representations can play an interesting role in non-Abelian quantum kinematics and dynamics. Certainly, the local substitution (3.9) brings the quantum kinematic formalism very close to the formalism of gauge field theory (cf. Chaichian and Nelipa, 1984).

4. THE GENERATING WAVE FUNCTIONS $\langle x|q\rangle_L$

We are now ready to examine the main idea of configuration ray representation theory. Namely, assuming the (rigged) Hilbert space $\mathscr{H}(G)$, let us discuss the possibility of having a family of kets $|x\rangle$ endowed with the fundamental ray property stated in equation (2.8). Thus we set, *ex hypothesis*

$$|x; k\rangle = \int d\mu_L(q)\, \psi^*_k(x; q)|q\rangle_L \tag{4.1}$$

where we define the wave function

$$\psi_k(x; q) = \langle x; k|q\rangle_L \tag{4.2}$$

on $X \times M(G)$. The continuous basis $\{|q\rangle_L\}$ is orthogonal and complete in $\mathscr{H}(G)$ with respect to the left invariant measure of G in $M(G)$. Henceforth, we write $|x;k\rangle$ instead of $|x\rangle$ [as we did in equation (2.8) for short], because these vectors depend on the ray constants. In this paper we are interested only in the *existence* of such kets $|x;k\rangle$ within the rigged Hilbert space attached with the carrier space $\mathscr{H}(G)$ of the regular representation. Here we shall not dwell on their physical applications (Krause, 1993*c*).

Upon substitution of equation (4.1) into (2.8), a straightforward calculation yields the following functional relation:

$$\psi_k[x; g(\bar{q}; q')]\, e^{-\phi_k[q;g(\bar{q};q')]} = \psi_k[f(x; q); q']\, e^{-i\varphi_k(x;q)} \tag{4.3}$$

which must be satisfied by ψ_k for all $q, q' \in M(G)$ and all $x \in X$. Hence, if one sets $q' = q$ in this relation, one obtains

$$\psi_k[f(x; q); q] = \psi_k(x; e)\, e^{i\varphi_k(x;q)} \tag{4.4}$$

from which

$$\psi_k(x; q) = \xi_k[f(x; \bar{q})] \exp\{i\varphi_k[f(x; \bar{q}); q]\} \tag{4.5}$$

follows. Here we have defined the *generating wave function* $\xi_k(x)$ of the configuration ray representation at the identity point $e \in M(G)$:

$$\xi_k(x) = \langle x; k|e\rangle_L \tag{4.6}$$

Equation (4.5) is the basic result. In particular, if one adopts the μ-gauge [i.e., equation (2.11)] (as one does in the applications), then equation (4.5) reads

$$\psi_k(x; q) = \xi_k[f(x; \bar{q})] \exp[-i\varphi_k(x; \bar{q})] \tag{4.7}$$

So, *given an allowable phase function* $\varphi_k(x; q)$ *and any generating wave function* $\xi_k(x)$, *one can construct a configuration ray representation of G within* $\mathscr{H}(G)$. This representation is carried by vectors of the general form

$$|x; k\rangle = \int d\mu_L(q)\, \xi_k^*[f(x; \bar{q})] \exp\{-i\varphi_k[f(x; \bar{q}); q]\}|q\rangle_L \tag{4.8}$$

The generating function $\xi_k(x)$ is completely arbitrary [provided $|x;k\rangle \in \mathscr{H}(G)$]. Indeed, it is a straightforward matter to show that *all* kets $|x;k\rangle$ of the form (4.8), where $\varphi_k(q;x)$ is an allowable phase function and $\xi_k(x)$ is a given function, will automatically satisfy equation (2.8). On the other hand, if one takes the derivative limits $\lim_{q\to e}\partial_a$ and $\lim_{q'\to e}\partial'_a$ in the functional relation (4.3), one obtains two sets of linear partial differential equations for the wave function $\psi_k(x;q)$. Then, after some standard manipulations of non-Abelian calculus, one shows that the most general solution to these equations is precisely of the form stated in equation (4.5) with $\xi_k(x)$ an arbitrary function of x. We here omit the proof of this theorem for the sake of briefness.

Hence, it is well established that one has a set of kets $|x; k\rangle$ belonging to the (rigged) Hilbert space of the regular representation of G, each carrying a configuration ray representation of the group. Since the generating function $\xi_k(x)$ remains arbitrary, the formalism has a *debarré d'excess* (as it must indeed, because thus far the analysis has been purely kinematic). In the applications of non-Abelian quantum kinematics to dynamics, one determines $\xi_k(x)$ on physical grounds, by means of some suitable superselection rules (cf. Krause, 1993*c*).

5. MISCELLANEOUS EXAMPLES

In this section we present two examples of phase calculus on $X \times M(G)$, which correspond to the following groups: (a) the Galilei group in one-dimensional space, and (b) the Newtonian group of the simple harmonic oscillator.

The physical applications of these examples have already been considered in two previous papers, but their detailed construction was omitted there (Krause, 1986, 1988).

5.1. The Galilei Group in One-Dimensional Space

Let us consider the Galilei group in one-dimensional space:

$$t' = t + q^1, \qquad x' = x + q^2 - q^3 t \tag{5.1}$$

whose infinitesimal operators are given by

$$Z_1(t, x) = \partial_t, \qquad Z_2(t, x) = \partial_x, \qquad Z_3(t, x) = -t\partial_x \tag{5.2}$$

Hence the well-known nilpotent Lie algebra follows:

$$[Z_1, Z_2] = 0, \qquad [Z_1, Z_3] = -Z_2, \qquad [Z_2, Z_3] = 0 \tag{5.3}$$

We next apply "phase calculus" to this particular realization of the group. To obtain a set of phase generators, one has to solve the system of equations (2.15), which now becomes

$$\sigma^{(k)}_{1,x} - \sigma^{(k)}_{2,t} = k_{12}, \qquad -t\sigma^{(k)}_{2,x} - \sigma^{(k)}_{3,x} = k_{23} \tag{5.4}$$

Indeed, as it follows immediately from the Lie algebra (5.3), of the three possible ray constants $\{k_{12}, k_{23}, k_{13}\}$, k_{13} is necessarily trivial. The most general solution to equations (5.4) can be written as

$$\sigma_1^{(k)}(t, x) = \frac{1}{2} k_{12} x + u(t, x)$$

$$\sigma_2^{(k)}(t, x) = -\frac{1}{2} k_{12} t + \int_0^t dt' \, u_{,x}(t', x) + v(x) \tag{5.5}$$

$$\sigma_3^{(k)}(t, x) = -k_{23} x - t \int_0^t dt' \, u_{,x}(t', x) - tv(x) + C$$

where $u(t, x)$ and $v(x)$ are arbitrary functions and C is an arbitrary constant. Therefore, if one performs a local $U(1)$ transformation, generated by

$$\lambda(t, x) = \int^t dt' \, u(t', x) + \int^x dx' \, v(x') \tag{5.6}$$

one reduces this solution to

$$\sigma_1^{(k)}(t. x) = \frac{1}{2} k_{12} x, \quad \sigma_2^{(k)}(t, x) = -\frac{1}{2} k_{12} t, \quad \sigma_3^{(k)}(t, x) = -k_{23} x + C \tag{5.7}$$

We then consider equation (2.13), for obtaining the phase function $\varphi_k(t, x; q)$ associated with the 2-cocycle

$$\phi_k(q'; q) = \frac{1}{2} k_{12} [q'^1(q^2 - q'^3 q^1) - q'^2 q^1] + \frac{1}{2} k_{23} [q'^2 q^3 - q'^3(q^2 + q^1 q^3)] \tag{5.8}$$

(cf. paper I), which belongs to the gauge $\mu(q) = 0$. It is a simple exercise to show that, because of the λ-transformation generated by $\lambda(q) = Cq^3$, one can take $C = 0$ in equation (5.7) without loss of generality. Hence, using $X_a(q)$ and $r_a^{(k)}(q)$ as given in paper I, we conclude that the system (2.13) now becomes

$$\partial_1 \varphi_k(t, x; q) = \frac{1}{2} k_{12}(x - q^3 t)$$

$$\partial_2 \varphi_k(t, x; q) = -\frac{1}{2} k_{12} t - \frac{1}{2} k_{23} q^3 \tag{5.9}$$

$$\partial_3 \varphi_k(t, x; q) = -\frac{1}{2} k_{12} q^1 t - k_{23}\left(x - q^3 t + \frac{1}{2} q^2\right)$$

which we readily integrate with the initial condition $\varphi_k(t, x; 0, 0, 0) = 0$. Thus we obtain the configuration phase function

$$\varphi_k(t, x; q^1, q^2, q^3) = \frac{1}{2} k_{12} [q^1 x - (q^2 + q^1 q^3) t] - \frac{1}{2} k_{23} q^3 (2x + q^2 - q^3 t) \tag{5.10}$$

which belongs to the μ-gauge. This phase function of the Galilei group was used to obtain the spacetime propagator kernel of a scalar Newtonian free particle by purely group-theoretic arguments (Krause, 1988).

It is clear that one may also consider the Galilei group as an *active* symmetry group which transforms one worldline of a free particle into another, i.e.,

$$x(t) = \alpha t + \beta \leftrightarrow x'(t') = \alpha' t + \beta' \tag{5.11}$$

One easily proves that under a Galilei transformation the *classical state* (α, β) of the system transforms into a new state (α', β') given by

$$\alpha' = \alpha + q^3, \qquad \beta' = \beta - \alpha q^1 + (q^2 - q^1 q^3) \tag{5.12}$$

This point transformation in *classical state space* $\{(\alpha, \beta)\}$ sets a new realization of the Galilei group, for which one may also calculate a phase function $\varphi_k(\alpha, \beta; q)$ as well as representation kets $|\alpha, \beta\rangle$ according to the techniques presented in this paper. Such a realization of the group in classical state space (i.e., *not* precisely the phase space) gives rise to a complementary ray representation of much physical interest from the standpoint of quantum kinematics. This subject shall be considered elsewhere.

5.2. The Newtonian Group of the Simple Harmonic Oscillator

We next consider the Newtonian point symmetry group attached to the equation of motion $\ddot{x} + \omega^2 x = 0$; namely

$$t' = t + q^1, \qquad x' = x + q^2 \cos \omega t + q^3 \sin \omega t \tag{5.13}$$

The infinitesimal operators of this realization are given by

$$Z_1(t, x) = \partial_t, \qquad Z_2(t, x) = (\cos \omega t)\partial_x, \qquad Z_3(t, x) = (\sin \omega t)\partial x \tag{5.14}$$

and the Lie algebra is as follows:

$$[Z_1, Z_2] = -\omega Z_3, \qquad [Z_1, Z_3] = \omega Z_2, \qquad [Z_2, Z_3] = 0 \tag{5.15}$$

Thus, this group has just one genuine ray constant: $k_{23} \neq 0$. Therefore, the differential equations (2.15) for the phase generators read

$$\begin{aligned} &(\sigma_2^{(k)} + i\sigma_3^{(k)})_{,t} - i\omega(\sigma_2^{(k)} + i\sigma_3^{(k)}) = e^{i\omega t}\omega_{1,x}^{(k)} \\ &(\sigma_3^{(k)} \cos \omega t - \sigma_2^{(k)} \sin \omega t)_{,x} = k \end{aligned} \tag{5.16}$$

with $k = -k_{23}$. After performing a suitable local $U(1)$ transformation, the general solution to these equations can be cast in the form

$$\sigma_1^{(k)}(t, x) = 0, \qquad \sigma_2^{(k)}(t, x) = -kx \sin \omega t, \qquad \sigma_3^{(k)}(t, x) = kx \cos \omega t \tag{5.17}$$

We then use these phase generators to obtain a configuration phase function according to equation (2.13). Therefore we consider the following 2-cocycle:

$$\begin{aligned}\phi_k(q'; q) = &\frac{1}{4} k[(q'^2)^2 + (q'^3)^2][\text{tg}\, \omega(q'^1 + q^1) - \text{tg}\, \omega q'^1] \\ &+ \frac{1}{4} k[(q^2)^2 + (q^3)^2][\text{tg}\, \omega(q'^1 + q^1) - \text{tg}\, \omega q^1] \\ &+ \frac{1}{2} k(q'^2 q^2 + q'^3 q^3)[\cos \omega q^1\, \text{tg}\, \omega(q'^1 + q^1) - \sin \omega q^1] \\ &+ \frac{1}{2} k(q'^3 q^2 - q'^2 q^3)[\sin \omega q^1\, \text{tg}\, \omega(q'^1 + q^1) + \cos \omega q^1] \end{aligned} \tag{5.18}$$

already obtained in paper I, which belongs to the μ-gauge, and whose right phase generators are given by

$$\begin{aligned} r_1^{(k)}(q) &= \frac{1}{4} k\omega[(q^2)^2 + (q^3)^2] \sec^2 \omega q^1 \\ r_2^{(k)}(q) &= -\frac{1}{2} kq^3 \sec \omega q^1 \\ r_3^{(k)}(q) &= \frac{1}{2} kq^2 \sec \omega q^1 \end{aligned} \tag{5.19}$$

The Lie operators $X_a(q)$ of the group are as follows:

$$\begin{aligned} &X_1 = \partial_1, \qquad X_2 = (\cos \omega q^1)\partial_2 - (\sin \omega q^1)\partial_3, \\ &X_3 = (\sin \omega q^1)\partial_2 + (\cos \omega q^1)\partial_3 \end{aligned} \tag{5.20}$$

Hence the equations we have to solve are of the form [cf. equation (2.13)]

$$\partial_1 \varphi_k(t, x; q) = -\frac{1}{4} k\omega[(q^2)^2 + (q^3)^3] \sec^2 \omega q^1 \tag{5.21a}$$

$$[(\cos \omega q^1)\partial_2 - (\sin \omega q^1)\partial_3]\varphi_k(t, x; q) = -k(x + q^2 \cos \omega t + q^3 \sin \omega t) \sin \omega(t + q^1) + \frac{k}{2} q^3 \sec \omega q^1 \tag{5.21b}$$

$$[(\sin \omega q^1)\partial_2 + (\cos \omega q^1)\partial_3]\varphi_k(t, x; q) = k(x + q^2 \cos \omega t + q^3 \sin \omega t) \cos \omega(t + q^1) - \frac{k}{2} q^2 \sec \omega q^1 \tag{5.21c}$$

One integrates equation (5.21a) immediately:

$$\varphi_k(t, x; q) = -\frac{k}{4}[(q^2)^2 + (q^3)^2] \text{ tg } \omega q^1 + u(t, x; q^2, q^3) \qquad (5.22)$$

while equations (5.21b) and (5.21c) can be written in the form

$$(\partial_2 + i\partial_3)\varphi_k(t, x; q)$$
$$= ik(x + q^2 \cos \omega t + q^3 \sin \omega t)\, e^{i\omega t} - i\frac{k}{2}(q^2 + iq^3) \sec \omega q^1\, e^{-i\omega q^1} \qquad (5.23)$$

Then a rather lengthy (but easy) process of integration, taking into account the initial condition (2.9), yields the desired answer:

$$\varphi_k(t, x; q^1, q^2, q^3) = -\frac{k}{4}[(q^2)^2 + (q^3)^2] \text{ tg } \omega q^1 - \frac{k}{4}[(q^2)^2 - (q^3)^2] \sin 2\omega t$$
$$+ \frac{k}{2} q^2 q^3 \cos 2\omega t - kx(q^2 \sin \omega t - q^3 \cos \omega t) \qquad (5.24)$$

which belongs to the μ-gauge indeed. One may check this solution against the 2-point functional relation (2.10) quite directly, with ϕ_k given in (5.18).

In another paper we have studied the quantum-kinematic model of the simple harmonic oscillator obtained from its Newtonian symmetry group. The regular ray representation, together with the configuration ray representation, of the Newtonian symmetry group of the system are enough to deduce the usual quantum mechanical model of the simple harmonic oscillator. In particular, the spacetime propagator kernel was obtained by means of this technique. No canonical quantization was used to this end (Krause, 1986).

Once again, here we can also treat an *active* symmetry group which transforms one wordline of the system into another, i.e.,

$$x(t) = \alpha \cos \omega t + \beta \sin \omega t \leftrightarrow x'(t') = \alpha' \cos \omega t' + \beta' \sin \omega t' \qquad (5.25)$$

Indeed, after some simple manipulations, one obtains the following realization of the group in classical state space $\{(\alpha, \beta)\}$:

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} \cos \omega q^1 & -\sin \omega q^1 \\ \sin \omega q^1 & \cos \omega q^1 \end{bmatrix} \begin{bmatrix} \alpha + q^2 \\ \beta + q^3 \end{bmatrix} \qquad (5.26)$$

(By the way, we observe that the Newtonian spacetime group of the simple harmonic oscillator is a realization of the two-dimensional Euclidean group E_2 of the plane.) The (α, β) realization of the group introduces another complementary ray representation for the description of the system; i.e., one also has kets $|\alpha, \beta\rangle \in \mathscr{H}(G)$ such that

$$U_L^{(k)}(q)|\alpha, \beta\rangle = e^{i\varphi_k(\alpha, \beta; q)}|\alpha', \beta'\rangle \qquad (5.27)$$

where (α', β') are given in equation (5.26). The phase function $\varphi_k(\alpha, \beta; q)$ as well as the kets $|\alpha, \beta\rangle$ are well determined by the method presented in this paper. The relation of these ("as classical as possible") states $|\alpha, \beta\rangle$ of quantum kinematics with the familiar coherent states of the harmonic oscillator will be discussed elsewhere.

6. CONCLUDING REMARKS

The techniques presented in this paper are well suited for the purposes of quantum kinematics. Some comments on this subject seem to be in order since we are now in possession of a broader perspective from which to understand this matter.

As the reader may have observed, all quantum-kinematic manipulations look quite familiar. Nevertheless, one must bear in mind the very meaning of the configuration representation kets $|x\rangle$, otherwise one could miss the point. Let us write them more explicitly: $|x\rangle = |t, \mathbf{x}\rangle$ [cf. equation (2.5) with $t = x^0$ and $\mathbf{x} = (x^1, \ldots, x^n)$]. In order to clarify this point, we have to recall that in quantum mechanics one sets the wave function $\psi_{QM}(t, \mathbf{x}) = \langle \mathbf{x}|\psi; t\rangle$ instead of $\psi_{QK}(t, \mathbf{x}) = \langle t, \mathbf{x}|\psi\rangle$ as one does in quantum kinematics. In fact, ordinary quantum mechanics rests on a *time evolution law*,

$$|\psi; t'\rangle = \exp[-(i/\hbar)(t' - t)H]|\psi; t\rangle \tag{6.1}$$

while quantum kinematics assumes a kind of *holistic evolution law* in X over the whole group manifold [cf. equation (2.8)]:

$$|t', \mathbf{x}'\rangle = |f^0(t, \mathbf{x}; q), \mathbf{f}(t, \mathbf{x}; q)\rangle = \exp[-i\varphi_k(t, \mathbf{x}; q)]U_L^{(k)}(q)|t, \mathbf{x}\rangle \tag{6.2}$$

which takes into account *all* the point symmetries of the system. Hence, the usual formalism of quantum mechanics is rather different from the generalized quantum-kinematic approach, for it treats time as a c-number while it quantizes $\mathbf{x}$, contrary to the relativistic requirement. On the other hand, quantum kinematics is intrinsically 'relativistic' in a very broad sense indeed (Mariwalla, 1975). It could be said that non-Abelian quantum kinematics lies somehow midway between quantum mechanics and quantum field theory. [For instance, in quantum kinematics one treats the carrier space $\tilde{\mathscr{H}}(G)$ as an *incoherent* rigged Hilbert space.]

Furthermore, in the application of non-Abelian quantum kinematics to dynamics, one identifies the allowed *physical* configuration spacetime kets $|t, \mathbf{x}\rangle$ by means of *super selection rules* dictated by the quantum-kinematic invariant operators (Krause, 1991). Then, after finding the *physical* generating wave function $\xi_k(t, \mathbf{x}) = \langle t, \mathbf{x}|e\rangle_L$ one calculates the transition amplitude:

$$\langle t, \mathbf{x}|t', \mathbf{x}'\rangle = \int d\mu_L(q)\, \xi_k[f(t, \mathbf{x}; \bar{q})\xi_k^*[f(t', \mathbf{x}'; \bar{q})] \times \exp\{-i[\varphi_k(t, \mathbf{x}; \bar{q}) - \varphi_k(t', \mathbf{x}'; \bar{q})]\} \tag{6.3}$$

which corresponds precisely to the configuration spacetime propagator kernel of the system. In fact, in a recent paper (Krause, 1993*c*) we have shown that in order to satisfy the superselection rules, the physical generating function $\xi_k(x)$ has to be a solution of a system of *fundamental eigenvalue wave equations*, which one has to solve in configuration spacetime in order to get the desired dynamical model. These are coupled eigenvalue wave equations of the general form

$$S_\alpha\left\{-i\hbar\left[Z_a(x) - i\sigma_a^{(k)}(x) - \frac{1}{2}f_{ab}^b\right]\right\}\xi_k(x; \epsilon) = \epsilon_\alpha \xi_k(x; \epsilon) \tag{6.4}$$

One gets such equations for each compatible superselection rule $(\alpha = 1, \ldots, s < r)$ and one solves them simultaneously. We refer to (6.4) as the *generalized Schrödinger equations* of quantum kinematics. The S_α are functions of the quantum-kinematic invariant operators. Their functional form is known because some of them are Casimir operators of the Lie algebra, while other are some of the quantum-kinematic invariant operators $R_a^{(k)}$ themselves. Of course, they all commute among themselves, and furthermore they commute with all the generators of the left regular (ray) representation.

Indeed, in the non-Abelian quantum-kinematic approach to dynamics, these generalized Schrödinger equations play the central role in the theory, in the same sense as the usual Schrödinger equation plays the central role in ordinary quantum mechanics. In particular, for an *isolated system*, time translation is a symmetry operation and therefore the usual Schrödinger equation will come automatically into the fore as *one* of the eigenvalue equations belonging to the system (6.4) of generalized wave equations. This means that the known functional form of the corresponding superselection operator, say $S_0 = S_0(R^{(k)})$, allows *the explicit calculation of the Hamiltonian operator* within the quantum kinematic model itself, as one foregoes the use of a 'gedanken' classical analog.

Note that there is really no transition $\epsilon_{(1)} \to \epsilon_{(2)}$ between the eigenvalues ϵ_α, for in evaluating the left-invariant integral (6.3) a multiple delta function $\delta^{(s)}(\epsilon_{(2)}; \epsilon_{(1)})$ is factorized out. (One gets Dirac deltas for the continuous spectra and Kronecker deltas for the discrete ones as necessary consequences of the superselection rules.) Therefore, the Hurwitz invariant integral (6.3) (covering the whole group manifold) corresponds to the *propagator kernel* in configuration spacetime X associated with the fundamental system of generalized Schrödinger wave equations (6.4). [This approach was used successfully in Krause (1986, 1988).] Work is in progress on this group-theoretic approach to quantum kinematics and dynamics.

Hence, concerning the general mathematical techniques for settling configuration ray representations of noncompact Lie groups, we deem our task as already complete. In many aspects, it seems that quantum kinematics comes very close to affording a new quantum mechanical theory that generalizes the standard theory of quantum mechanics. Such a theory might provide more insight into the role that symmetry plays in quantum physics. However, we do not know whether such a generalized quantum mechanics is possible at all. If it is possible, it would be an important theoretical achievement, especially in the realm of high-energy physics. Much remains to be done in this intriguing area, which we deem as deserving further formal research as well as more applications.

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